Math 1020C Week 2

Basic Algebra

Let $a, b \in \mathbb{R}$

$$(a \pm b)^2 = a^2 \pm 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

$$a^2 - b^2 = (a+b)(a-b)$$

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

$$a^n - b^n = (a-b)(a^n + a^{n-1}b + \cdots + ab^{n-1} + b^n)$$

$$\sqrt{a^2} = |a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

Exponents and Radical

$$a^n = \underbrace{a \cdot a \cdot a \cdot \cdots \cdot a}_{n\ a\text{'s}} \qquad a = \text{base}, \quad n = \text{exponent}$$

$$a^0 = 1 \qquad a^{-n} = \frac{1}{a^n}$$

$$a^{\frac{1}{n}} = \sqrt[n]{a} = b \Rightarrow a = b^n$$

eg $\quad 2^3 = 8 \qquad 8^{\frac{1}{3}} = 2$

Rmk If n is even, $a^{\frac{1}{n}} = \sqrt[n]{a}$ is defined only for $a \geq 0$

$$\boxed{\begin{array}{ll} a^m \cdot a^n = a^{m+n} & a^m / a^n = a^{m-n} \\ a^m b^m = (ab)^m & a^m / b^m = (a/b)^m \\ (a^m)^n = a^{mn} & \end{array}}$$

Rationalization (Get rid of $\sqrt{\ }$)

$$\frac{a}{b} \begin{array}{l} \leftarrow \text{numerator} \\ \leftarrow \text{denominator} \end{array}$$

Rationalize Denominator

eg. $\dfrac{\sqrt{2}}{\sqrt{3}} = \dfrac{\sqrt{2}}{\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{\sqrt{6}}{3}$

eg. $\dfrac{4-\sqrt{3}}{5+\sqrt{3}} = \dfrac{4-\sqrt{3}}{5+\sqrt{3}} \cdot \dfrac{5-\sqrt{3}}{5-\sqrt{3}}$

$$= \frac{20-9\sqrt{3}+3}{5^2-(\sqrt{3})^2}$$

$$= \frac{23-9\sqrt{3}}{22}$$

Rationalize Numerator

eg. $\dfrac{\sqrt{2}}{\sqrt{3}} = \dfrac{\sqrt{2}}{\sqrt{3}} \cdot \dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{2}{\sqrt{6}}$

$$\frac{1+\sqrt{x}}{1-x} = \frac{1+\sqrt{x}}{1-x} \cdot \frac{1-\sqrt{x}}{1-\sqrt{x}}$$

$$= \frac{1-(\sqrt{x})^2}{(1-x)(1-\sqrt{x})} = \frac{1}{1-\sqrt{x}}$$

$$\sqrt{x^2+4x}-x = \frac{\sqrt{x^2+4x}-x}{1} \cdot \frac{\sqrt{x^2+4x}+x}{\sqrt{x^2+4x}+x}$$

$$= \frac{x^2+4x-x^2}{\sqrt{x^2+4x}+x}$$

$$= \frac{4x}{\sqrt{x^2+4x}+x}$$

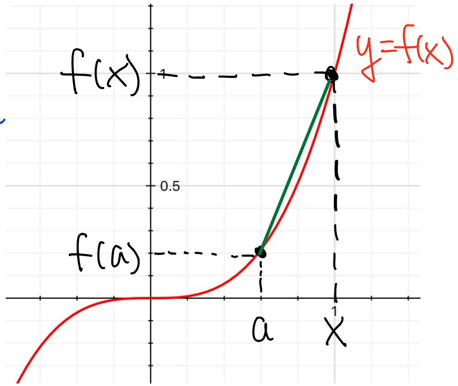
Difference Quotient

eg Let $f(x) = x^3$ $\quad g(x) = \dfrac{1}{\sqrt{x+1}}$.

Simplify the following difference quotients

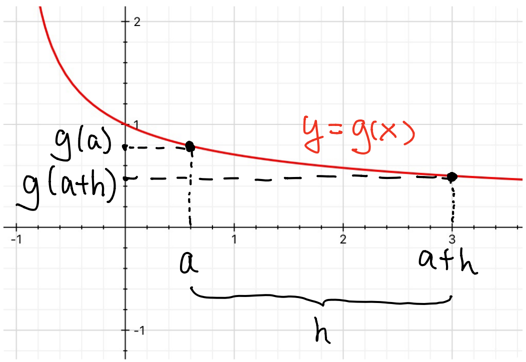
i
$$\frac{f(x)-f(a)}{x-a} = \frac{x^3-a^3}{x-a}$$
$$= \frac{(x-a)(x^2+ax+a^2)}{x-a}$$
$$= x^2+ax+a^2$$

slope of the line joining $(x, f(x))$ and $(a, f(a))$



ii
$$\frac{g(a+h)-g(a)}{h} = \frac{\frac{1}{\sqrt{a+h+1}} - \frac{1}{\sqrt{a+1}}}{h}$$
$$= \frac{\sqrt{a+1}-\sqrt{a+h+1}}{h\cdot\sqrt{a+h+1}\cdot\sqrt{a+1}} \cdot \frac{\sqrt{a+1}+\sqrt{a+h+1}}{\sqrt{a+1}+\sqrt{a+h+1}}$$
$$= \frac{(a+1)-(a+h+1)}{h\cdot\sqrt{a+h+1}\cdot\sqrt{a+1}\,(\sqrt{a+1}+\sqrt{a+h+1})}$$
$$= \frac{-h}{h\cdot\sqrt{a+h+1}\cdot\sqrt{a+1}\,(\sqrt{a+1}+\sqrt{a+h+1})}$$
$$= \frac{-1}{\sqrt{a+h+1}\cdot\sqrt{a+1}\,(\sqrt{a+1}+\sqrt{a+h+1})}$$

Better for taking limit $h \to 0$



Some important Sets

$\mathbb{N}$ = the set of all natural numbers

$= \{1, 2, 3, 4, 5, \cdots\}$

$\mathbb{Z}$ = the set of all integers

$= \{\cdots -3, -2, -1, 0, 1, 2, 3, \cdots\}$

$= \{0, \pm 1, \pm 2, \pm 3, \pm 4, \cdots\}$

$\mathbb{Q}$ = the set of all rational numbers

$\left\{ \frac{m}{n} : m, n \in \mathbb{Z} \text{ and } n \neq 0 \right\}$

$\mathbb{R}$ = the set of all real numbers

eg $\frac{2}{3} \in \mathbb{Q}$ but $\frac{2}{3} \notin \mathbb{Z}$

$\sqrt{2} \in \mathbb{R}$, $\sqrt{2} \notin \mathbb{Q}$

$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$

Intervals

Let $a, b \in \mathbb{R}$ or $\pm\infty$

Open interval (Endpoints not included)

$(a, b) = \{x \in \mathbb{R} : a < x < b\}$

Closed interval (Endpoints included)

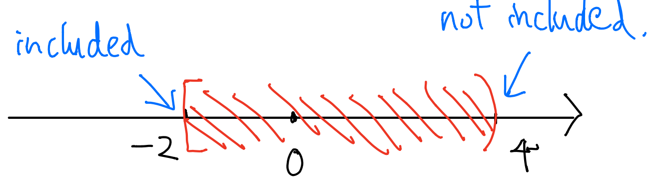
$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$

Half-open interval

$(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$

$[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$

Picture $[-2, 4)$



Function

Let A, B be sets

A function $f: A \to B$ is a rule of assigning to each element of A an element of B

A = Domain of f (Set of inputs)

B = Codomain of f (A set containing all outputs)

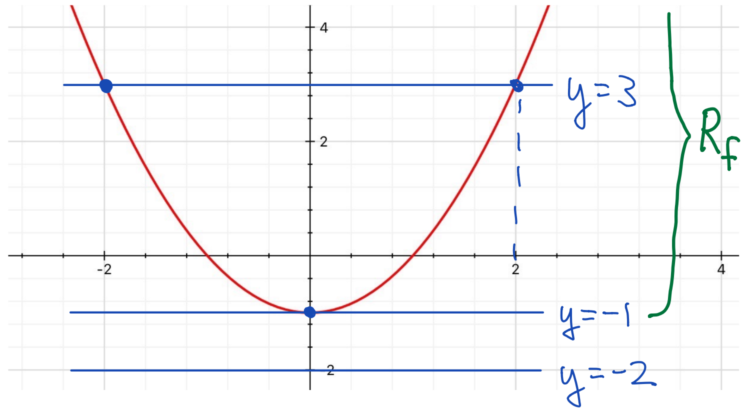
R_f = Range of f (Set of outputs)

$= \{f(x) \in B : x \in A\}$

Other notations

D_f = domain of f

eg $\underbrace{f: \mathbb{R}}_{\text{domain}} \to \underbrace{\mathbb{R}}_{\text{codomain}}$ with $\underbrace{f(x) = x^2 - 1}_{\text{"rule of assignment"}}$



(no intersection)

$f(0) = -1 \Rightarrow -1 \in R_f$

$f(2) = 3 \Rightarrow 3 \in R_f$

$f(x) \neq -2$ for any $x \in D_f$

$\because x^2 \geq 0$

$\therefore f(x) = x^2 - 1 \geq -1$ $\Rightarrow -2 \notin R_f$

$R_f = [-1, \infty)$

Implied domain

If a function $f(x)$ is given by an expression without specifying its domain, then the domain will be assumed to be the largest subset of $\mathbb{R}$ such that the expression makes sense.

That domain is called the Implied domain (or natural domain)

Useful rules

① Denominator $\neq 0$

② For $\log g(x)$, need $g(x) > 0$

③ Let m be an positive even number

For $\sqrt[m]{h(x)} = [h(x)]^{\frac{1}{m}}$,

need $h(x) \geq 0$

Rmk For ③,

eg $m = 3$ (odd)

$64^{\frac{1}{3}} = 4$

$(-64)^{\frac{1}{3}} = -4$

No problem

eg $m = 4$ (even)

$(64)^{\frac{1}{4}} = 8^{\frac{1}{2}}$

$(-64)^{\frac{1}{4}}$ is not real !

(fourth root of negative number)

eg Find implied domain of

a. $\log(x^2 - 3x - 10)$

b. $\dfrac{x-3}{\sqrt[4]{3-|x|}}$

c. $(x+2)^{\frac{2}{3}}$

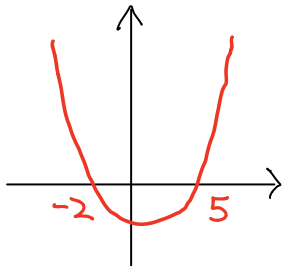
d. $f-g$, where $f(x) = \dfrac{1}{1+x}$ $g(x) = \dfrac{1}{1-x}$

Sol

a. Need $x^2 - 3x - 10 > 0$

$(x-5)(x+2) > 0$

$\therefore \ x > 5$ or $x < -2$



$\Rightarrow$ Implied domain $= (-\infty, -2) \cup (5, \infty)$

b. Need $3 - |x| \geq 0$ under $\sqrt[4]{}$

Also, $\sqrt[4]{3-|x|} \neq 0$

$\Rightarrow \quad 3 - |x| > 0$

$\Rightarrow \quad 3 > |x|$

$\Rightarrow \quad -3 < x < 3$

Implied domain $= (-3, 3)$

c. $(x+2)^{\frac{2}{3}} = \sqrt[3]{(x+2)^2}$

3 is odd!

Addition, square, cubic root are defined for any real numbers

Implied domain $= \mathbb{R} = (-\infty, \infty)$

d. $(f-g)(x) = f(x) - g(x)$

$$= \underbrace{\frac{1}{1+x}}_{x \neq -1} - \underbrace{\frac{1}{1-x}}_{x \neq 1}$$

$\Rightarrow$ Implied domain $= \mathbb{R} \setminus \{\pm 1\}$

$= (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

Operations on functions

Let $f(x)$, $g(x)$ be functions. Define

$$(f \pm g)(x) = f(x) \pm g(x)$$

$$(fg)(x) = f(x)\,g(x)$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad \text{if } g(x) \neq 0$$

$$(g\circ f)(x) = g(f(x)) \qquad \text{(Composition)}$$

$$x \xmapsto{f} f(x) \xmapsto{g} (g\circ f)(x)$$

$$\boxed{\begin{aligned} &D_{f+g} = D_{f-g} = D_{fg} = D_f \cap D_g \\ &D_{f/g} = (D_f \cap D_g) \setminus \{x \in D_g : g(x) = 0\} \\ &D_{g\circ f} = \{x \in D_f : f(x) \in D_g\} \end{aligned}}$$

eg Let $f(x) = x^2 - x$, $g: (2, \infty) \to \mathbb{R}$

a. Find $(f\circ f)(3)$.

b. Find the implied domain of $g\circ f$.

Sol

a. $(f\circ f)(3) = f(f(3)) = f(6) = 30$

b. $g\circ f(x) = g(f(x)) = g(x^2 - x)$

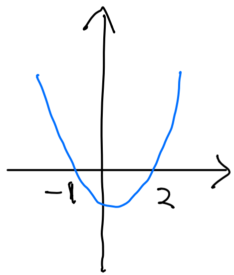
$D_g = (2, \infty) \Rightarrow x^2 - x \in (2, \infty)$

$\Rightarrow x^2 - x > 2$

$\Rightarrow x^2 - x - 2 > 0$

$\Rightarrow (x-2)(x+1) > 0$

$\Rightarrow x > 2$ or $x < -1$



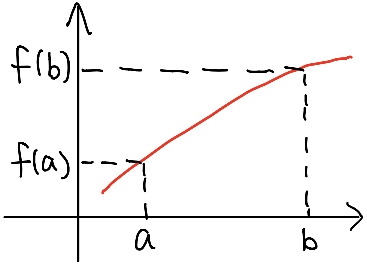
$D_{g\circ f} = (-\infty, -1) \cup (2, \infty)$

Increasing / Decreasing functions

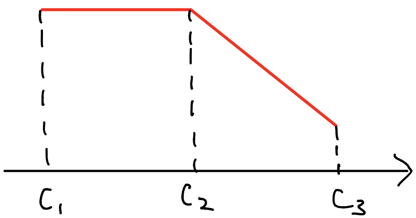
Let I be an interval. A function

$f(x)$ is said to be $\begin{cases} \text{increasing} \\ \text{strictly increasing} \\ \text{decreasing} \\ \text{strictly decreasing} \end{cases}$ on I

if $\begin{cases} f(a) \le f(b) \\ f(a) < f(b) \\ f(a) \ge f(b) \\ f(a) > f(b) \end{cases}$ for any $a < b$ on I



strictly increasing



decreasing on $[c_1, c_3]$

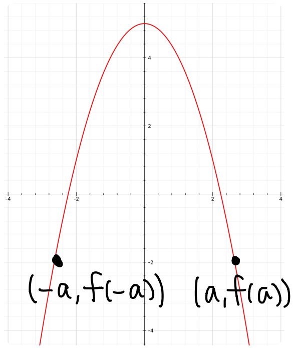
strictly decreasing on $[c_2, c_3]$

Even / Odd functions

Def If $f(-x) = f(x)$ for any $x \in D_f$, then $f(x)$ is called an <u>even function</u>

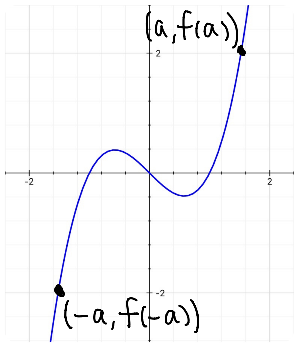
If $f(-x) = -f(x)$ for any $x \in D_f$, then $f(x)$ is called an <u>odd function</u>

Even function



Symmetric about y-axis

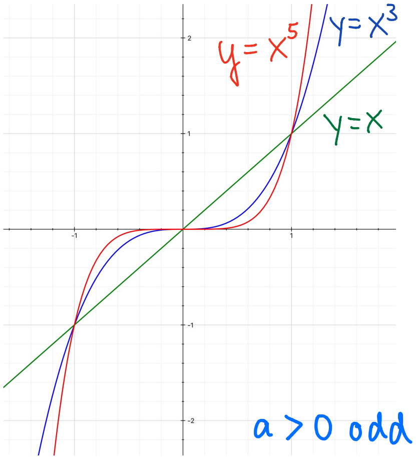
Odd function



Symmetric about origin.

More example of functions

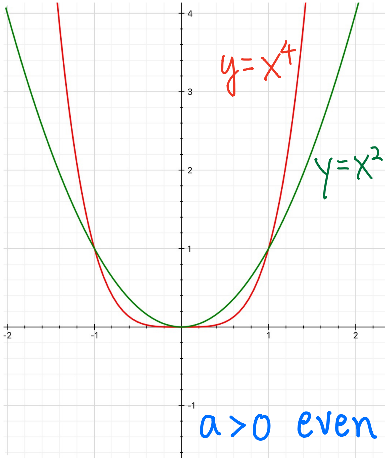
Power functions $f(x) = x^a$



Domain $= \mathbb{R} = (-\infty, \infty)$

Odd function

strictly increasing on $(-\infty, \infty)$

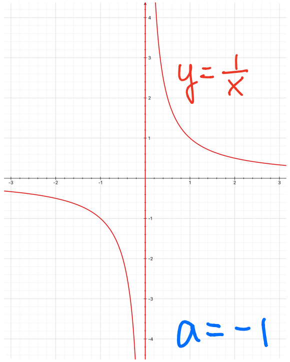


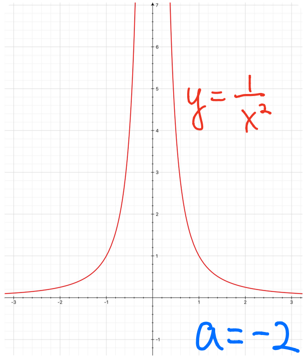
Domain $= \mathbb{R} = (-\infty, \infty)$

Even function

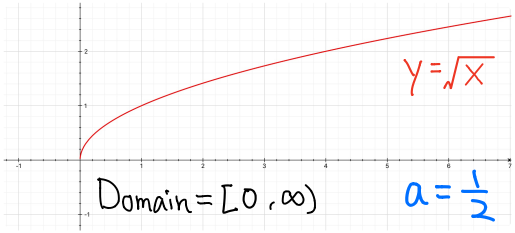
strictly increasing on $[0, \infty)$

strictly decreasing on $(-\infty, 0]$

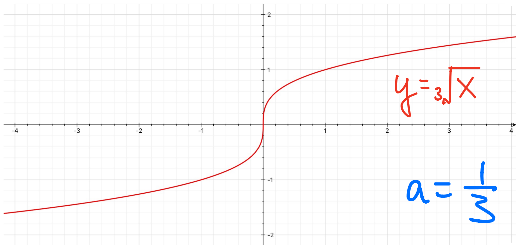




Domain $= \mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$



Domain $= [0, \infty)$



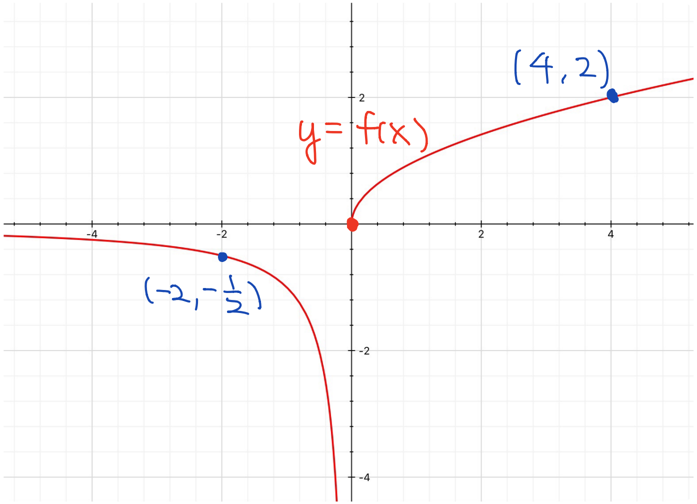
Domain $= \mathbb{R} = (-\infty, \infty)$

Piecewise Functions

eg

$$f(x) = \begin{cases} \sqrt{x} & \text{if } x \geq 0 \\ \frac{1}{x} & \text{if } x < 0 \end{cases}$$

- $f(4) = \sqrt{4} = 2$ $\quad (\because 4 \geq 0)$
- $f(-2) = \frac{1}{-2} = -\frac{1}{2}$ $\quad (\because -2 < 0)$



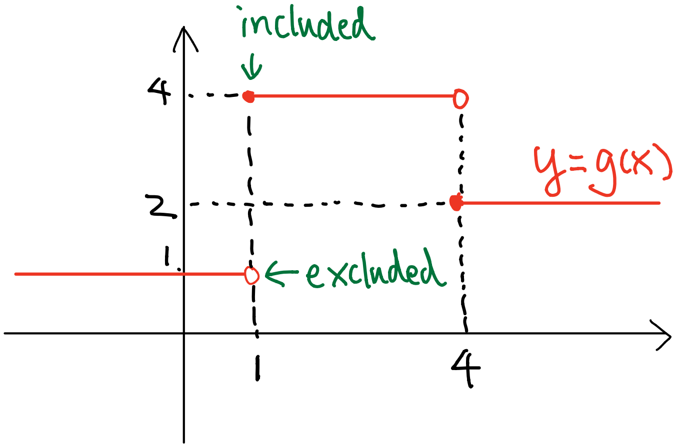
Q If $|h| < 1$, $f(1+h) = ?$

Note $-1 < h < 1 \Rightarrow 0 < 1+h < 2$

$\therefore f(1+h) = \sqrt{1+h}$

eg

$$g(x) = \begin{cases} 1 & \text{if } x < 1 \\ 4 & \text{if } 1 \leq x < 4 \\ 2 & \text{if } x \geq 4 \end{cases}$$

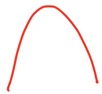


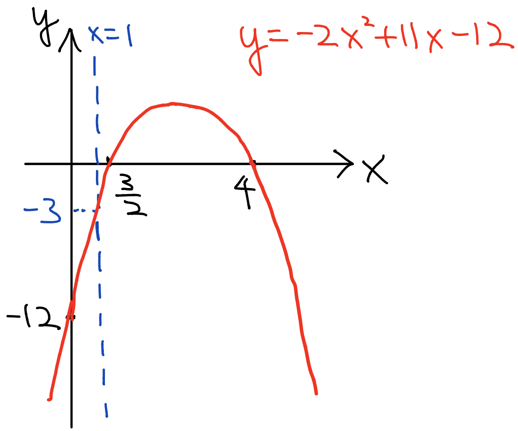
e.g. Graph $h(x)=\begin{cases} 2x+3 & \text{if } x<1 \\ -2x^2+11x-12 & \text{if } x\geq 1 \end{cases}$

Sol Note that if $-2x^2+11x-12=0$

then $x=\dfrac{-11\pm\sqrt{11^2-4(-2)(-12)}}{2(-2)}$

$=\dfrac{-11\pm\sqrt{25}}{-4}=\dfrac{3}{2}$ or 4

Also, leading coefficient $=-2<0 \Rightarrow$ 

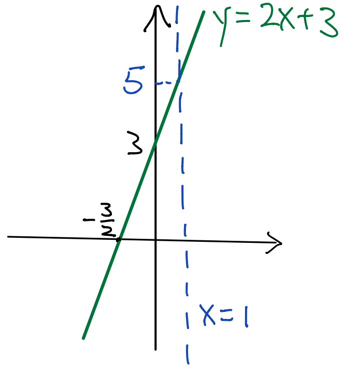


For the graph $y=2x+3$,

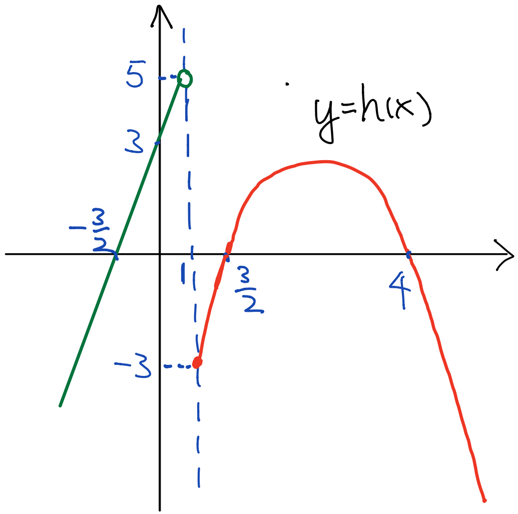
Slope $=2$

y-intercept $=3$

x-intercept $=-\frac{3}{2}$



$\therefore$ Graph of $h(x)$:

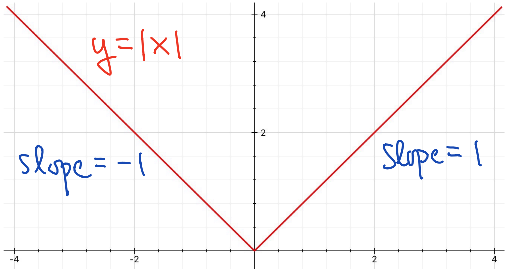


Absolute Value

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

eg. $3 \geq 0 \Rightarrow |3| = 3$

$-2 < 0 \Rightarrow |-2| = -(-2) = 2$



Prop For $x \in \mathbb{R}$,

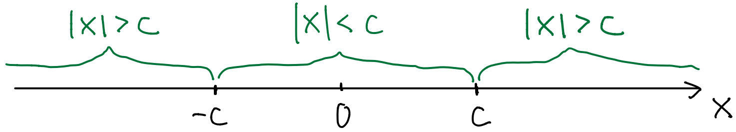
- $|x| \geq 0$
- $|x| = |-x|$
- $|x| = \sqrt{x^2}$
- $|x|^2 = x^2$
- $|xy| = |x||y|$
- $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$
- $|x+y| \leq |x| + |y|$ if $x, y \in \mathbb{R}$

Prop Let $c > 0$. Then

① $|f(x)| < c \Leftrightarrow -c < f(x) < c$

② $|f(x)| > c \Leftrightarrow f(x) > c$ or $f(x) < -c$

③ Similar statements for $\leq$ and $\geq$



eg. Solve ① $|2x-3| \leq 7$ ② $|3x+2| > 4$

Sol

① $|2x-3| \leq 7$

$\Rightarrow -7 \leq 2x-3 \leq 7$

$\Rightarrow -4 \leq 2x \leq 10$

$\Rightarrow -2 \leq x \leq 5$

② $|3x+2| > 4$

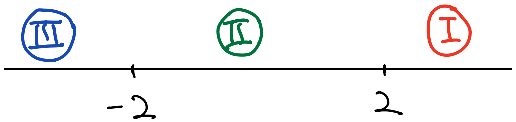
$\Rightarrow 3x+2 > 4$ or $3x+2 < -4$

$\Rightarrow 3x > 2$ or $3x < -6$

$\Rightarrow x > \frac{2}{3}$ or $x < -2$

eg Graph $f(x) = |x-2| + |x+2|$

<u>Sol</u> Consider 3 cases



<u>Case I</u> If $x \geq 2$,

then $x-2 \geq 0$, $x+2 \geq 0$

$$\begin{aligned} f(x) &= |x-2| + |x+2| \\ &= (x-2) + (x+2) \\ &= 2x \end{aligned}$$

<u>Case II</u> If $-2 \leq x < 2$

then $x-2 < 0$, $x+2 \geq 0$

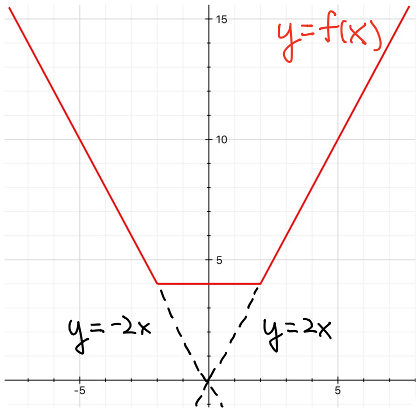
$$\begin{aligned} f(x) &= |x-2| + |x+2| \\ &= -(x-2) + (x+2) \\ &= 4 \end{aligned}$$

<u>Case III</u> If $x < -2$,

then $x-2 < 0$, $x+2 < 0$

$$\begin{aligned} f(x) &= |x-2| + |x+2| \\ &= -(x-2) - (x+2) \\ &= -2x \end{aligned}$$

$$\therefore f(x) = \begin{cases} 2x & \text{if } x \geq 2 \\ 4 & \text{if } -2 \leq x < 2 \\ -2x & \text{if } x < -2 \end{cases}$$



Inequality

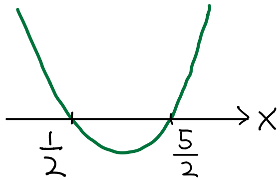
a. $4x^2 + 5 \geqslant 12x$

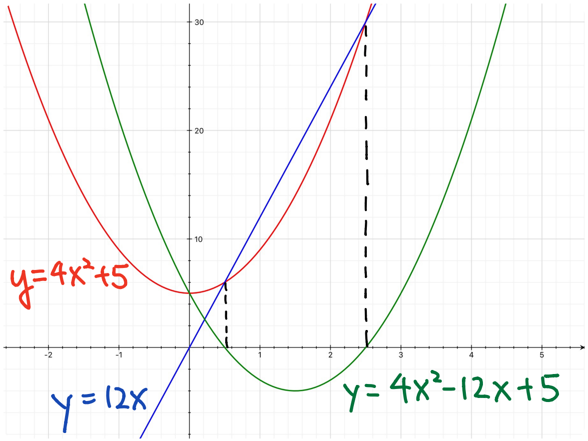
Sol

$$4x^2 - 12x + 5 \geqslant 0$$

$$(2x-1)(2x-5) \geqslant 0$$

$$\therefore\ x \leq \frac{1}{2} \quad \text{or} \quad x \geqslant \frac{5}{2}$$





b. $\dfrac{2x-1}{x+1} < 1$

WRONG approach

$$\frac{2x-1}{x+1} < 1$$

Multiply both sides by $x+1$ ⊛

$$\Rightarrow \quad 2x - 1 < x + 1$$

$$x < 2$$

Why WRONG? It is because if $x+1 < 0$, the step ⊛ reverses the inequality

Prop Let $a, b, c \in \mathbb{R}$, $a > b$

① If $c > 0$, then $ca > cb$

② If $c < 0$, then $ca < cb$

③ Similar statements for $a \leq b$

Correct approach 1

$$\frac{2x-1}{x+1} < 1$$

Note that $x+1 \neq 0 \Rightarrow (x+1)^2 > 0$

Multiply both sides by $(x+1)^2$

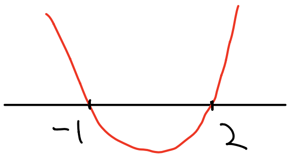
$$\Rightarrow \quad (2x-1)(x+1) < (x+1)^2$$

$$(2x-1)(x+1) - (x+1)^2 < 0$$

$$(2x-1-x-1)(x+1) < 0$$

$$(x-2)(x+1) < 0$$

$$-1 < x < 2$$



Correct approach 2

$$\frac{2x-1}{x+1} < 1$$

$$\frac{2x-1}{x+1} - 1 < 0$$

$$\frac{2x-1-(x+1)}{x+1} < 0$$

$$\frac{x-2}{x+1} < 0$$

Consider 3 cases

	$x < -1$	$-1 < x < 2$	$x > 2$
$x-2$	$-$	$-$	$+$
$x+1$	$-$	$+$	$+$
$\frac{x-2}{x+1}$	$+$	$-$	$+$

$\therefore \ -1 < x < 2$

c. $x - \dfrac{3}{x} \geq 2$

<u>Sol</u>

$$x - \frac{3}{x} - 2 \geq 0$$

$$\frac{x^2 - 3 - 2x}{x} \geq 0$$

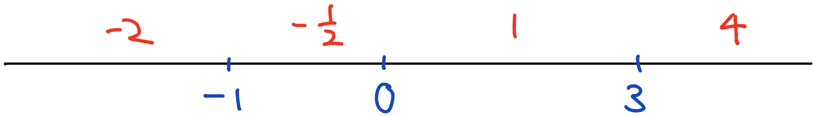
$$\frac{(x-3)(x+1)}{x} \geq 0$$

The points $-1, 0, 3$ divide $(-\infty, \infty)$ into 4 intervals

	$x<-1$	$x=-1$	$-1<x<0$	$x=0$	$0<x<3$	$x=3$	$x>3$
$x-3$	$-$	$-$	$-$	$-$	$-$	0	$+$
$x+1$	$-$	0	$+$	$+$	$+$	$+$	$+$
x	$-$	$-$	$-$	0	$+$	$+$	$+$
$\dfrac{(x-3)(x+1)}{x}$	$-$	0	$+$	undefined	$-$	0	$+$

Hence, $-1 \leq x < 0$ or $x \geq 3$

<u>Rmk</u> One may also determine the sign on each interval by testing with a point on that interval :



For example, let $g(x) = x - \dfrac{3}{x} - 2$

$$g(-2) = -\frac{5}{2} < 0 \Rightarrow g(x) < 0 \text{ on } (-\infty, -1)$$

$$g(-\tfrac{1}{2}) = \frac{7}{2} > 0 \Rightarrow g(x) > 0 \text{ on } (-1, 0)$$